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# Evaluation of Integrals and Sums Involving $[\sin(Mx)/\sin(x)]^n$

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## Preface

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## LIST OF SYMBOLS

$M$	integer, number of elements in line array
$g(t)$	arbitrary function of $t$ , (1)
$G(\omega)$	Fourier transform of $g(t)$ , (1)
$V$	integral of product of $g(t) h^*(t)$ , (2)
$H(\omega)$	Fourier transform of $h(t)$ , (2)
$\gamma$	argument of sine functions, (3)
$h_1(t)$	ratio of sine functions, (3)
prime	sum with prime denotes every other term, (3)
$H_1(\omega)$	Fourier transform of $h_1(t)$ , (4)
$V_1$	general integral (5)
$h_2(t)$	square of ratio of sine functions, (7)
$H_2(\omega)$	Fourier transform of $h_2(t)$ , (8)
$V_2$	general integral (9)
$\phi(m)$	autocorrelation of weights, (12B)
sub $r$	real part, (13B)
$V_{1a}$	integral (18)
$V_{2a}$	integral (22)
$V_{1b}$	integral (27)
$V_{2b}$	integral (28)
$V_{1c}$	integral (33)
$V_{2c}$	integral (34)
$\Delta$	increment in $t$ , (41), (A-1)
$\delta_\Delta(t)$	impulse train, (41)
$\otimes$	convolution, (42)
$S_N(M, k)$	discrete sine ratio, (52)

INT	integer part, (55)
$V_5$	general integral (63)
$\psi(p)$	autocorrelation of $\phi(m)$ , (61)
$g(n\Delta)$	samples of $g(t)$ , (A-1)
$\underline{G}(\omega)$	approximation to $G(\omega)$ , (A-1)
N	size of FFT, (A-6)
$g_c(n\Delta)$	collapsed version of $g(n\Delta)$ , (A-7)
$\tilde{g}(n\Delta)$	phase modulated $g(n\Delta)$ , (A-11)
$\tilde{g}_c(n\Delta)$	collapsed version of $\tilde{g}(n\Delta)$ , (A-12)

EVALUATION OF INTEGRALS AND SUMS  
INVOLVING  $[\sin(Mx)/\sin(x)]^n$

## INTRODUCTION

The response of an equiweighted equispaced line array to a distributed field involves the kernel  $\sin(Mx)/\sin(x)$  or its square, depending on whether the voltage or power response, respectively, is of interest [1;2]. Numerical evaluation of such integrals can be very time consuming for two reasons: this kernel oscillates quickly with  $x$  for large  $M$ , and it does not decay with  $x$ . This necessitates fine sampling in  $x$  and large integration regions, both of which can lead to a significant computational burden, especially for two-dimensional or three-dimensional arrays. The object of this report is to give an alternative numerical procedure that can be very advantageous in some cases, and, in fact, leads to closed forms for some examples.

The procedure is also applied to summations involving the same kernel. Its utility depends on the rate of decay of the complementary part of the original integrand, as compared with the Fourier transform of this component. In any event, an alternative is presented for the user to consider in any numerical investigation.



## GENERAL APPROACH

For arbitrary function  $g(t)$ , define its Fourier transform as

$$G(\omega) = \int dt \exp(-i\omega t) g(t) . \quad (1)$$

(Integrals without limits are over the range of nonzero integrand.) Then Parseval's theorem states that the following two alternative integrals are equal:

$$V = \int dt g(t) h^*(t) = \frac{1}{2\pi} \int d\omega G(\omega) H^*(\omega) . \quad (2)$$

Here,  $H(\omega)$  is the Fourier transform of  $h(t)$ . Now, if  $H(\omega)$  takes on a noticeably simpler form than  $h(t)$ , then the second integral in (2) can offer an attractive alternative to the first integral in (2). That will indeed be the case here.

## CASE 1

For integer  $M \geq 1$  and constant  $\gamma > 0$ , consider the special choice of  $h(t)$  as

$$\begin{aligned} h_1(t) &= \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{n=0}^{M-1} \exp[i\gamma t(2n+1-M)] = \\ &= \sum_{m=1-M}^{M-1}{}' \exp(i\gamma t m) , \end{aligned} \quad (3)$$

where the prime on the latter sum denotes skipping every other term. Then the Fourier transform, according to (1), is

$$H_1(\omega) = 2\pi \sum_{m=1-M}^{M-1}{}' \delta(\omega - \gamma m) . \quad (4)$$

Substitution of (3) and (4) in (2) yields

$$V_1 = \int dt \, g(t) \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1}{}' G(\gamma m) . \quad (5)$$

This result indicates that if  $G(\omega)$ , the Fourier transform of  $g(t)$ , can be evaluated, then the  $t$  integral in (5) is given by a finite sum of equispaced samples of  $G(\omega)$  at increment  $2\gamma$ . The (complex) function  $g(t)$  in (5) is arbitrary, except that the integral must converge. When  $G(\omega)$  cannot be analytically evaluated, then proper application of a fast Fourier transform procedure to  $g(t)$  can be tailored to yield precisely the equispaced samples required for the right-hand side of (5); this technique is detailed in appendix A.

If function  $G(\omega)$  is even in  $\omega$ , then (5) simplifies to

$$V_1 = \left\{ \begin{array}{ll} 2 \sum_{m=1}^{M-1} G(\gamma m) & \text{for } M = 2, 4, 6, \dots \\ G(0) + 2 \sum_{m=2}^{M-1} G(\gamma m) & \text{for } M = 1, 3, 5, \dots \end{array} \right\}. \quad (6)$$

A program for (6) is given in appendix B.

## CASE 2

For integer  $M \geq 1$  and constant  $\gamma > 0$ , consider the alternative special choice of  $h(t)$  as

$$h_2(t) = \left[ \frac{\sin(M\gamma t)}{\sin(\gamma t)} \right]^2 = \sum_{n,k=0}^{M-1} \exp[i\gamma t(2n-2k)] = \quad (7A)$$

$$= \sum_{m=1-M}^{M-1} (M - |m|) \exp(i2\gamma t m), \quad (7B)$$

where we used (3). There is no prime on the latter summation because all terms from  $1-M$  to  $M-1$  are to be included. The Fourier transform of  $h_2(t)$  is

$$H_2(\omega) = 2\pi \sum_{m=1-M}^{M-1} (M - |m|) \delta(\omega - 2\gamma m). \quad (8)$$

The use of (7A) and (8) in (2) yields

$$V_2 = \int dt \, g(t) \left[ \frac{\sin(My t)}{\sin(\gamma t)} \right]^2 = \sum_{m=1-M}^{M-1} (M - |m|) G(2\gamma m) . \quad (9)$$

Again, the integral of interest is given by a finite sum of samples of the Fourier transform of  $g(t)$ , also at increment  $2\gamma$  in  $\omega$ . The fast Fourier transform technique discussed in appendix A is relevant here also. If  $G(\omega)$  is even in  $\omega$ , then we can express (9) as

$$V_2 = M G(0) + 2 \sum_{m=1}^{M-1} (M - m) G(2\gamma m) \quad \text{for all } M \geq 1 . \quad (10)$$

A program for (10) is given in appendix B.

### CASE 3

For arbitrary weights  $\{w_m\}$  and frequencies  $\{\gamma_m\}$ , with

$$h_3(t) = \sum_m w_m \exp(i\gamma_m t) , \quad (11A)$$

then we have a generalization of (3), with

$$H_3(\omega) = 2\pi \sum_m w_m \delta(\omega - \gamma_m) . \quad (11B)$$

(Summations without limits are over the range of nonzero summand.) Use of these expressions in general result (2) yields

$$V_3 = \int dt \, g(t) \sum_m w_m^* \exp(-i\gamma_m t) = \sum_m w_m^* G(\gamma_m) . \quad (11C)$$

Again, the Fourier transform of  $g(t)$  is required, but now at general arguments  $\{\gamma_m\}$ .

## CASE 4

Function  $h_2(t)$  in (7) is a special case of the weighted array power response

$$h_4(t) = \left| \sum_k w_k \exp(-i2\gamma tk) \right|^2 = \sum_m \phi(m) \exp(-i2\gamma tm) , \quad (12A)$$

where  $\phi(m)$  is the autocorrelation of the weights:

$$\phi(m) = \sum_k w_k w_{k-m}^* = \phi^*(-m) . \quad (12B)$$

The integral in (9) is then generalized to

$$\begin{aligned} V_4 &= \int dt g(t) h_4^*(t) = \int dt g(t) \left| \sum_k w_k \exp(-i2\gamma tk) \right|^2 = \\ &= \sum_m \phi(m) G(2\gamma m) , \end{aligned} \quad (13A)$$

upon use of (12A), where  $g(t)$  can be complex and nonsymmetric. Thus, integral  $V_4$  requires the autocorrelation of weights  $\{w_k\}$  and the Fourier transform of  $g(t)$  for its evaluation. The earlier result in (9) corresponds to weights  $w_k = 1$  for  $1 \leq k \leq M$ .

When function  $g(t)$  is real (but possibly nonsymmetric) and the weights are real, (13A) can be simplified to

$$V_4 = \phi(0) G_r(0) + 2 \sum_{m \geq 1} \phi(m) G_r(2\gamma m) , \quad (13B)$$

where  $G_r(\omega)$  is the real part of Fourier transform  $G(\omega)$  in (1).

A program for (13B) is given in appendix B.

## EXAMPLES

## EXAMPLE A

The first example of interest is

$$g_a(t) = \frac{1}{(t-\mu)^2 + \beta^2}, \quad \beta > 0. \quad (14)$$

Its Fourier transform is

$$G_a(\omega) = \frac{\pi}{\beta} \exp(-i\mu\omega - \beta|\omega|), \quad (15)$$

for which the real part is

$$G_{ar}(\omega) = \frac{\pi}{\beta} \cos(\mu\omega) \exp(-\beta|\omega|). \quad (16)$$

Since integral (5) is obviously real for example (14), we obtain

$$v_{1a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(Myt)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1} G_{ar}(\gamma m). \quad (17)$$

Substitution of (16) in (17) yields the closed form result

$$v_{1a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(Myt)}{\sin(\gamma t)} = \frac{2\pi}{\beta D} \left[ E_{M+3} C_{M-1} - E_{M+1} C_{M+1} + \begin{cases} C_1 E_1 (1 - E_2) & \text{for } M \text{ even} \\ \frac{1}{2} (1 - E_4) & \text{for } M \text{ odd} \end{cases} \right], \quad (18)$$

where

$$E_m = \exp(-\beta\gamma m), \quad C_m = \cos(\mu\gamma m), \quad D = 1 - 2 E_2 C_2 + E_4. \quad (19)$$

A program for (18) and (19) follows; it is written in BASIC for the Hewlett Packard 9000 computer.

```

10 INPUT M,Beta,Gamma,Mu      ! Beta > 0, Gamma > 0
20 B=Beta*Gamma
30 C=Mu*Gamma
40 E=EXP(-B*2)
50 IF (M MODULO 2)=1 THEN 80
60 F=COS(C)*SQR(E)*(1-E)
70 GOTO 90
80 F=.5-.5*E*E
90 A=E*COS(C*(M-1))-COS(C*(M+1))
100 A=A*EXP(-B*(M+1))+F
110 V1a=A*2*PI/(Beta*(1-2*E*COS(C*2)+E*E))
120 PRINT V1a
130 END

```

When we instead substitute (14) and (16) in (9), there follows

$$\begin{aligned}
 v_{2a} &= \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(Myt)}{\sin(\gamma t)} \right]^2 = \\
 &= \frac{\pi}{\beta} \sum_{m=1-M}^{M-1} (M - |m|) \cos(2\mu\gamma m) \exp(-2\beta\gamma|m|) . \quad (20)
 \end{aligned}$$

This finite sum can be written in compact form by use of [3; 0.113]. Namely, define here

$$\begin{aligned}
 E &= \exp(-2\beta\gamma) , & C &= \cos(2\mu\gamma) , & S &= \sin(2\mu\gamma) , \\
 E_M &= \exp(-2\beta\gamma M) , & C_M &= \cos(2\mu\gamma M) , & S_M &= \sin(2\mu\gamma M) , \\
 A &= 1 - E^2 , & B &= 1 + E^2 , & D &= B - 2 E C . \quad (21)
 \end{aligned}$$

Then



$$v_{2a} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(M\gamma t)}{\sin(\gamma t)} \right]^2 =$$

$$= \frac{2\pi}{\beta D} \left[ \frac{M}{2} A - \frac{E}{D} \left( (C B - 2 E)(1 - E_M C_M) + S A E_M S_M \right) \right] . \quad (22)$$

A program for (21) and (22) follows.

```

10  INPUT M,Beta,Gamma,Mu      ! Beta > 0,  Gamma > 0
20  Tb=2*Beta*Gamma
30  Tm=2*Mu*Gamma
40  E=EXP(-Tb)
50  A=E*E
60  B=1+A
70  A=1-A
80  C=COS(Tm)
90  D=B-2*E*C
100 Em=EXP(-Tb*M)
110 T=(C*B-2*E)*(1-Em*COS(Tm*M))
120 T=T+SIN(Tm)*A*Em*SIN(Tm*M)
130 T=.5*M*A-T*E/D
140 V2a=T*2*PI/(Beta*D)
150 PRINT V2a
160 END

```

#### EXAMPLE B

The next example to be considered is

$$g_b(t) = \frac{1}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} , \quad \beta > 0 , \quad \alpha > 0 . \quad (23)$$

Since  $g_b(t)$  is a product of two functions, its Fourier transform  $G_b(\omega)$  is given by a convolution of the individual transforms.

The Fourier transform of the first term in (23) has already been encountered in (15), and the Fourier transform of the second term in (23) is a rectangle located on interval  $(-\alpha, \alpha)$  in  $\omega$ .

Therefore,  $G_b(\omega)$  is given by convolution

$$G_b(\omega) = \frac{\pi}{2\alpha\beta} \int_{\omega-\alpha}^{\omega+\alpha} du \exp(-i\mu u - \beta|u|) . \quad (24)$$

Since  $g_b(t)$  in (23) is real, we need only evaluate the real part,  $G_{br}(\omega)$ , of  $G_b(\omega)$ . With the aid of auxiliary variables

$$\begin{aligned} C_\omega &= \cos(\mu\omega), \quad S_\omega = \sin(\mu\omega), \quad \underline{C}_\omega = \cosh(\beta\omega), \quad \underline{S}_\omega = \sinh(\beta\omega), \\ C_\alpha &= \cos(\mu\alpha), \quad S_\alpha = \sin(\mu\alpha), \quad \underline{C}_\alpha = \cosh(\beta\alpha), \quad \underline{S}_\alpha = \sinh(\beta\alpha), \\ B_1 &= \underline{C}_\omega C_\omega (\beta C_\alpha - \mu S_\alpha) + \underline{S}_\omega S_\omega (\mu C_\alpha + \beta S_\alpha), \\ B_2 &= \underline{S}_\alpha C_\alpha (\beta C_\omega - \mu S_\omega) + \underline{C}_\alpha S_\alpha (\mu C_\omega + \beta S_\omega), \end{aligned} \quad (25)$$

we find that  $G_{or}(\omega)$  is given by

$$G_{br}(\omega) = \frac{\pi}{\alpha\beta(\beta^2 + \mu^2)} \begin{cases} \beta - \exp(-\beta\alpha) B_1 & \text{for } 0 \leq \omega \leq \alpha \\ \exp(-\beta\omega) B_2 & \text{for } \alpha \leq \omega \end{cases} . \quad (26)$$

To complete the description, we observe that  $G_{br}(\omega)$  is even in  $\omega$  because  $g_b(t)$  is real. A program for  $G_{br}(\omega)$  follows, where we have made the following identifications:  $W \equiv \omega$ ,  $A \equiv \alpha$ ,  $B \equiv \beta$ ,  $U \equiv \mu$ .

```

10 DEF FNGbr(W,A,B,U)          100 IF Wa<A THEN 150
20 Wa=ABS(W)                   110 Ra=1./Ea
30 F=PI/(A*B*(B*B+U*U))        120 T=(Ra-Ea)*Ca*(B*Cw-U*Sw)
40 Ea=EXP(-B*A)                 130 B2=.5*(T+(Ra+Ea)*Sa*(U*Cw+B*Sw))
50 Ew=EXP(-B*Wa)                140 RETURN F*Ew*B2
60 Ca=COS(U*A)                  150 Rw=1./Ew
70 Cw=COS(U*Wa)                 160 T=(Rw+Ew)*Cw*(B*Ca-U*Sa)
80 Sa=SIN(U*A)                  170 B1=.5*(T+(Rw-Ew)*Sw*(U*Ca+B*Sa))
90 Sw=SIN(U*Wa)                 180 RETURN F*(B-Ea*B1)
                                190 FNEND

```

If we now employ (23) in (5), we obtain

$$v_{1b} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} \frac{\sin(My t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1} G_{br}(\gamma m), \quad (27)$$

where  $G_{br}(\omega)$  is given by (25), (26), and its even property. Since there is a break in the analytic form for  $G_{br}(\omega)$  at  $\omega = \pm\alpha$ , it is not reasonable to perform the summation in (27) in closed form; those terms in (27) for  $\gamma|m| \leq \alpha$  utilize the upper line of (26), while those for  $\gamma|m| \geq \alpha$  utilize the lower line of (26). However, since  $G_{br}(\omega)$  is even in  $\omega$ , the simplification in (6) is applicable.

Instead, when (23) is substituted in (9), there follows

$$v_{2b} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} \left[ \frac{\sin(My t)}{\sin(\gamma t)} \right]^2 = \sum_{m=1-M}^{M-1} (M - |m|) G_{br}(2\gamma m), \quad (28)$$

where  $G_{br}(\omega)$  is given by (25) and (26). Again, the break in form of  $G_{br}(\omega)$  at  $\omega = \pm\alpha$  precludes a closed form result for the summation in (28); also, the simplification in (10) is immediately applicable to (28).

## EXAMPLE C

The final example is

$$g_c(t) = \frac{1}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(\alpha t)}{\alpha t} \right]^2, \quad \beta > 0, \quad \alpha > 0. \quad (29)$$

The Fourier transform of the second term in (29) is a triangle located on interval  $(-2\alpha, 2\alpha)$  in  $\omega$ . Therefore,  $G_c(\omega)$  is given by convolution

$$G_c(\omega) = \frac{\pi}{2\alpha\beta} \int_{\omega-2\alpha}^{\omega+2\alpha} du \exp(-i\mu u - \beta|u|) \left( 1 - \frac{|\omega-u|}{2\alpha} \right). \quad (30)$$

Because  $g_c(t)$  is real, only the real part of (30) is needed. This tedious calculation has been carried through, with the following result; define auxiliary variables

$$\begin{aligned} R &= \beta^2 - \mu^2, \quad I = 2\beta\mu, \quad D = \beta^2 + \mu^2, \quad E_\omega = \exp(-\beta\omega), \quad E_\alpha = \exp(-2\beta\alpha), \\ \underline{C}_\alpha &= \cosh(2\beta\alpha), \quad \underline{S}_\alpha = \sinh(2\beta\alpha), \quad C_\alpha = \cos(2\mu\alpha), \quad S_\alpha = \sin(2\mu\alpha), \\ \underline{C}_\omega &= \cosh(\beta\omega), \quad \underline{S}_\omega = \sinh(\beta\omega), \quad C_\omega = \cos(\mu\omega), \quad S_\omega = \sin(\mu\omega), \\ C_1 &= \underline{C}_\omega C_\omega (R C_\alpha - I S_\alpha) + \underline{S}_\omega S_\omega (R S_\alpha + I C_\alpha), \\ C_2 &= \underline{C}_\alpha C_\alpha (R C_\omega - I S_\omega) + \underline{S}_\alpha S_\alpha (R S_\omega + I C_\omega), \\ C &= R C_\omega - I S_\omega. \end{aligned} \quad (31)$$

Then we find that

$$G_{c,c}(\omega) = \frac{\pi}{2\alpha^2\beta D^2} \left\{ \begin{aligned} &D \beta (2\alpha - \omega) - E_\omega C + E_\alpha C_1 && \text{for } 0 \leq \omega \leq 2\alpha \\ &- E_\omega C + E_\omega C_2 && \text{for } 2\alpha \leq \omega \end{aligned} \right\} \quad (32)$$

Also,  $G_{cr}(\omega)$  is even in  $\omega$ . A program for  $G_{cr}(\omega)$  is listed below, where  $W \equiv \omega$ ,  $A \equiv \alpha$ ,  $B \equiv \beta$ ,  $U \equiv \mu$ .

```

10 DEF FNGcr(W,A,B,U)      ! A > 0 , B > 0
20 Wa=ABS(W)
30 Tb=2.*A*B
40 Tu=2.*A*U
50 Bw=B*Wa
60 Uw=U*Wa
70 B2=B*B
80 U2=U*U
90 R=B2-U2
100 I=2.*B*U
110 D=B2+U2
120 Ew=EXP(-Bw)
130 Ea=EXP(-Tb)
140 Ca=COS(Tu)
150 Sa=SIN(Tu)
160 Cw=COS(Uw)
170 Sw=SIN(Uw)
180 C=R*Cw-I*Sw
190 IF Wa<2.*A THEN 250
200 Ra=1./Ea
210 C2=.5*(Ra+Ea)*Ca*C
220 C2=.5*(Ra-Ea)*Sa*(R*Sw+I*Cw)+C2
230 T=Ew*(C2-C)
240 GOTO 290
250 Rw=1./Ew
260 C1=.5*(Rw+Ew)*Cw*(R*Ca-I*Sa)
270 C1=.5*(Rw-Ew)*Sw*(R*Sa+I*Ca)+C1
280 T=D*(Tb-Bw)-Ew*C+Ea*C1
290 RETURN PI*T/(Tb*A*D*D)
300 FNEND

```

We now substitute (29) into (5) and get

$$v_{1c} = \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(\alpha t)}{\alpha t} \right]^2 \frac{\sin(Myt)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1} G_{cr}(\gamma m), \quad (33)$$

where  $G_{cr}(\omega)$  is given by (31), (32), and its even character. The break in form in (32) at  $\omega = \pm 2\alpha$  precludes a closed form for the sum in (33). However, (6) is still applicable.

When (29) is utilized in (9), there follows

$$\begin{aligned}
 v_{2c} &= \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(\alpha t)}{\alpha t} \right]^2 \left[ \frac{\sin(M\gamma t)}{\sin(\gamma t)} \right]^2 = \\
 &= \sum_{m=1-M}^{M-1} (M - |m|) G_{cr}(2\gamma m) .
 \end{aligned} \tag{34}$$

Equation (10) may also be employed here.

#### SPECIAL CASES

If we set  $M = 1$  in (17), there follows

$$\int \frac{dt}{(t-\mu)^2 + \beta^2} = G_{ar}(0) = \frac{\pi}{\beta} , \tag{35}$$

where we used (16). The same case in (27) yields

$$\begin{aligned}
 &\int \frac{dt}{(t-\mu)^2 + \beta^2} \frac{\sin(\alpha t)}{\alpha t} = G_{br}(0) = \\
 &= \frac{\pi}{\alpha\beta(\beta^2 + \mu^2)} \{ \beta - \exp(-\beta\alpha) [\beta \cos(\mu\alpha) - \mu \sin(\mu\alpha)] \} , \tag{36}
 \end{aligned}$$

upon use of (26) and (25). Finally, from (33),

$$\begin{aligned}
 & \int \frac{dt}{(t-\mu)^2 + \beta^2} \left[ \frac{\sin(\alpha t)}{\alpha t} \right]^2 = G_{cr}(0) = \\
 & = \frac{\pi}{2\alpha^2\beta(\beta^2 + \mu^2)^2} \left( 2\alpha\beta(\beta^2 + \mu^2) - R + E_{\alpha}(R C_{\alpha} - I S_{\alpha}) \right), \quad (37)
 \end{aligned}$$

using (32) and (31).

## APPLICATION TO SUMS

In this section, it is more convenient to use Parseval's theorem (2) in the form

$$v = \int dt \, g(t) \, h^*(t) = \int df \, G(f) \, H^*(f) , \quad (38)$$

where Fourier transform

$$G(f) = \int dt \, \exp(-i2\pi ft) \, g(t) . \quad (39)$$

Now, we take as our candidate  $h(t)$  function,

$$h(t) = p(t) \, \Delta \delta_{\Delta}(t) , \quad (40)$$

where  $\delta_{\Delta}(t)$  is the infinite impulse train

$$\delta_{\Delta}(t) = \sum_k \delta(t - k\Delta) . \quad (41)$$

The Fourier transform of  $h(t)$  is then

$$H(f) = P(f) \otimes \delta_{1/\Delta}(f) = \sum_k P\left(f - \frac{k}{\Delta}\right) , \quad (42)$$

where  $P(f)$  is the Fourier transform of  $p(t)$ ,  $\otimes$  denotes convolution, and we have utilized the fact that the Fourier transform of impulse train  $\Delta \delta_{\Delta}(t)$  is another impulse train,  $\delta_{1/\Delta}(f)$ .

Substitution of (40) and (42) in (38) yields



$$V = \Delta \sum_k g(k\Delta) p^*(k\Delta) = \sum_k \int df G(f) P^*\left(f - \frac{k}{\Delta}\right) . \quad (43)$$

For general  $p(t)$  and  $P(f)$ , this will not be a useful relation, since the right-hand side of (43) is an infinite sum of integrals. However, we will be interested here only in the special cases of

$$p(t) = \left[ \frac{\sin(My t)}{\sin(\gamma t)} \right]^n , \quad n \text{ integer} . \quad (44)$$

CASE  $n = 0$

For  $n = 0$ , the above relations specialize to

$$p(t) = 1 , \quad P(f) = \delta(f) ,$$

$$H(f) = \sum_k \delta\left(f - \frac{k}{\Delta}\right) ,$$

$$V_0 = \Delta \sum_k g(k\Delta) = \sum_k G\left(\frac{k}{\Delta}\right) . \quad (45)$$

This is a discrete version of Parseval's theorem. Although one infinite sum has been traded for another, we can now choose that alternative that has the most rapidly decaying (and/or easily computed) summand for numerical evaluation.

CASE  $n = 1$

Now we have, via (3),

$$p(t) = \frac{\sin(M\gamma t)}{\sin(\gamma t)} = \sum_{m=1-M}^{M-1} \exp(i\gamma t m) . \quad (46)$$

There follows

$$P(f) = \sum_{m=1-M}^{M-1} \delta\left(f - \frac{\gamma m}{2\pi}\right) ,$$

$$H(f) = \sum_k \sum_{m=1-M}^{M-1} \delta\left(f - \frac{k}{\Delta} - \frac{\gamma m}{2\pi}\right) ,$$

$$V_1 = \Delta \sum_k g(k\Delta) \frac{\sin(M\gamma\Delta k)}{\sin(\gamma\Delta k)} = \quad (47)$$

$$= \sum_k \sum_{m=1-M}^{M-1} G\left(\frac{k}{\Delta} + \frac{\gamma m}{2\pi}\right) . \quad (48)$$

Again, we have an alternative infinite sum (48) that hopefully decays faster than the original sum (47). The  $\sin(Mx)/\sin(x)$  term does not help convergence in (47) because this term never decays for large  $x$ . Although (48) is a double sum, the summation on  $m$  only contains  $M$  terms; the utility of (48) depends heavily on the asymptotic decay of  $G(f)$  for large  $f$ .

CASE n = 2

With the aid of (7), we now find

$$p(t) = \left[ \frac{\sin(M\gamma t)}{\sin(\gamma t)} \right]^2 = \sum_{m=1-M}^{M-1} (M - |m|) \exp(i2\gamma t m) ,$$

$$P(f) = \sum_{m=1-M}^{M-1} (M - |m|) \delta\left(f - \frac{\gamma m}{\pi}\right) ,$$

$$H(f) = \sum_k \sum_{m=1-M}^{M-1} (M - |m|) \delta\left(f - \frac{k}{\Delta} - \frac{\gamma m}{\pi}\right) ,$$

$$V_2 = \Delta \sum_k g(k\Delta) \left[ \frac{\sin(M\gamma\Delta k)}{\sin(\gamma\Delta k)} \right]^2 = \quad (49)$$

$$= \sum_k \sum_{m=1-M}^{M-1} (M - |m|) G\left(\frac{k}{\Delta} + \frac{\gamma m}{\pi}\right) . \quad (50)$$

## EXAMPLE

Consider, as in (14) and (16),

$$g_a(t) = \frac{1}{(t-\mu)^2 + \beta^2} ,$$

$$G_{ar}(f) = \frac{\pi}{\beta} \cos(2\pi\mu f) \exp(-2\pi\beta|f|) . \quad (51)$$

The summations in (47) and (49) are very slowly decaying, leading to difficulty in attaining accurate results. The alternatives in

(48) and (50), on the other hand, have exponential decay and can be evaluated quite accurately. The additional examples given earlier in (23)-(26) and in (29)-(32), along with the corresponding programs, lend reasonable alternatives to some otherwise lengthy numerical calculations.

### SOME RELATED SUMS

Here, we collect a few closed form results for sums involving the  $\sin(Mx)/\sin(x)$  kernel. For ease of notation, define

$$S_N(M, k) = \frac{\sin(Mk\pi/N)}{\sin(k\pi/N)} . \quad (52)$$

Observe that

$$S_N(M, k) = \begin{cases} M & \text{for } k = 0, \pm 2N, \pm 4N, \dots \\ M(-1)^{M-1} & \text{for } k = \pm N, \pm 3N, \dots \end{cases} . \quad (53)$$

Then, we find the sum over one interval to be

$$\sum_{k=0}^{N-1} S_N(M, k) = \begin{cases} M & \text{for } M \text{ even} \\ N(1 + 2J) & \text{for } M \text{ odd} \end{cases} , \quad (54)$$

where

$$J = \text{INT}\left(\frac{M-1}{2N}\right) . \quad (55)$$

The sum over a double interval is

$$\sum_{k=0}^{2N-1} S_N(M, k) = \begin{cases} 0 & \text{for } M = 0, 2, 4, \dots \\ 2N & \text{for } M = 1, 3, \dots, 2N-1 \end{cases} . \quad (56)$$

The correlation on the second variable of  $S_N$  is

$$\sum_{k=0}^{N-1} S_N(M, k) S_N(M, k+j) = N S_N(M, j) \quad \text{for } 0 \leq M \leq N \text{ and all } j. \quad (57)$$

Finally, the correlation on the first variable is

$$\begin{aligned} \sum_{k=0}^{N-1} S_N(M, k) S_N(M+2L, k) &= M(M + 2L) + \\ &+ \begin{cases} \underline{M}(N - \underline{M} - 2\underline{L}) & \text{for } 0 \leq \underline{M} + \underline{L} \leq N \\ N(3\underline{M} + 2\underline{L} - 2N) - \underline{M}(\underline{M} + 2\underline{L}) & \text{for } N < \underline{M} + \underline{L} \end{cases} , \quad (58) \end{aligned}$$

for all  $M, L, N$ , where

$$\underline{M} = M \bmod N , \quad \underline{L} = L \bmod N . \quad (59)$$

## SUMMARY

Extensions to integrals involving  $[\sin(Mx)/\sin(x)]^n$  for  $n > 2$  are possible, based upon the results presented here. For example, starting from (12A) for arbitrary weights, we could consider

$$h_5(t) = h_4^2(t) = \sum_p \psi(p) \exp(-i2\gamma tp) , \quad (60)$$

where

$$\psi(p) = \sum_m \phi(m) \phi^*(m-p) \quad (61)$$

is the autocorrelation of sequence  $\{\phi(m)\}$  defined in (12B).

Therefore, Fourier transform

$$H_5(\omega) = 2\pi \sum_p \psi(p) \delta(\omega + 2\gamma p) , \quad (62)$$

giving rise to

$$v_5 = \int dt g(t) h_5(t) = \sum_p \psi(p) G(2\gamma p) . \quad (63)$$

The case of equal weights  $\{w_k\}$  in (12A) now corresponds to  $n = 4$  in the sine function ratio above, and  $\psi(p)$  is the autocorrelation of a triangular sequence.

The evaluation of integrals and sums involving the term  $[\sin(Mx)/\sin(x)]^n$  can often be simplified by the use of Parseval's theorem because this term has a Fourier transform which is a finite sum of delta functions. Major effort can then be concentrated on getting the Fourier transform of the complementary part of the integrand. This procedure has been applied here to several examples which arise in evaluation of the response of equispaced arrays to distributed spatial fields. For more complicated fields, a fast Fourier transform procedure combined with the above result leads to a very efficient method of integral evaluations. Applications of this procedure have been made in [5].

## APPENDIX A - USE OF FAST FOURIER TRANSFORM

The summations for  $V_1$  and  $V_2$  in (5) and (9), respectively, require the evaluation of the Fourier transform of  $g(t)$ , namely  $G(\omega)$ , at equispaced increment  $2\gamma$ . But this latter function can be approximated by means of the trapezoidal rule according to

$$G(\omega) = \int dt \exp(-i\omega t) g(t) = \\ \approx \Delta \sum_n \exp(-i\omega \Delta n) g(n\Delta) \equiv \underline{G}(\omega) = \sum_n G\left(\omega - n\frac{2\pi}{\Delta}\right), \quad (A-1)$$

where  $\Delta$  is the sampling increment in  $t$ . The latter summation in (A-1) indicates aliasing lobes separated by  $2\pi/\Delta$  on the  $\omega$  axis. In order to control aliasing, we must choose  $\Delta$  small enough, say  $\Delta < \Delta_0$ . Then samples of approximation  $\underline{G}(\omega)$  in (A-1) at multiples of  $2\gamma$  are given by

$$\underline{G}(2\gamma m) = \Delta \sum_n \exp(-i2\gamma \Delta mn) g(n\Delta). \quad (A-2)$$

Now since  $\Delta$  is arbitrary, except for upper limit  $\Delta_0$ , choose

$$\Delta = \frac{\pi}{N\gamma}, \quad (A-3)$$

where  $N$  is an integer and  $\gamma$  is the prescribed increment in  $\omega$ . In order that  $\Delta$  be less than  $\Delta_0$ , we must take integer

$$N > \frac{\pi}{\gamma \Delta_0}. \quad (A-4)$$

Use of (A-3) in (A-2) gives



$$\underline{G}(2\gamma m) = \Delta \sum_n \exp(-i2\pi mn/N) g(n\Delta) = \quad (A-5)$$

$$= \Delta \sum_{n=0}^{N-1} \exp(-i2\pi mn/N) g_c(n\Delta) , \quad (A-6)$$

where "collapsed" sequence [4; pages 4-5]

$$g_c(n\Delta) = \sum_k g(n\Delta + kN\Delta) \quad \text{for } 0 \leq n \leq N - 1 . \quad (A-7)$$

The manipulation from (A-5) to (A-6) is exact; it avoids truncation error normally associated with functions  $g(t)$  which decay slowly with  $t$ . The sum on  $k$  in (A-7) must be carried out (for each  $n$ ) until negligible values for  $g$  are encountered for both positive as well as negative values of  $k$ .

Equation (A-6) indicates that values of  $\underline{G}(2\gamma m)$  for  $m = 0$  to  $N - 1$  are available by an  $N$ -point fast Fourier transform when  $N$  is a power of 2. Values for negative  $m$  are available in location  $m \bmod N$ . In order to get all the desired values of  $\underline{G}(2\gamma m)$  required for (9), without aliasing, we also require that  $N/2 > M$ . Thus, the final condition on integer  $N$  is

$$N > \max\left(\frac{\pi}{\gamma\Delta_0}, 2M\right) . \quad (A-8)$$

For the case of (5), where the increment on  $\omega$  is  $2\gamma$ , but starting at  $\omega = \gamma$ , we return to (A-1) to find that

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_n \exp(-i\gamma\Delta n - i2\gamma\Delta mn) g(n\Delta) . \quad (A-9)$$

The same choice of  $\Delta$  in (A-3) now yields

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_n \exp(-i2\pi mn/N) \exp(-i\pi n/N) g(n\Delta) . \quad (A-10)$$

This result is identical to (A-5) except that  $g(n\Delta)$  must be replaced by

$$\exp(-i\pi n/N) g(n\Delta) = \tilde{g}(n\Delta) . \quad (A-11)$$

Calculation of the collapsed version of  $\tilde{g}$  is eased by the observation that

$$\begin{aligned} \tilde{g}_c(n\Delta) &= \sum_k \tilde{g}(n\Delta + kN\Delta) = \\ &= \sum_k \exp(-i\pi(n + kN)/N) g(n\Delta + kN\Delta) = \\ &= \exp(-i\pi n/N) \sum_k (-1)^k g(n\Delta + kN\Delta) \quad \text{for } 0 \leq n \leq N-1 , \quad (A-12) \end{aligned}$$

thereby leading exactly to

$$\underline{G}(\gamma + 2\gamma m) = \Delta \sum_{n=0}^{N-1} \exp(-i2\pi mn/N) \tilde{g}_c(n\Delta) . \quad (A-13)$$

The leading phase factor in (A-12) only needs to be evaluated at  $N$  different values, and the sum in (A-12) requires differencing of "adjacent" samples of  $g$  spaced by  $N\Delta$ , rather than the straight summation previously adequate for (A-6) and (A-7). Condition (A-8) applies here as well.

## APPENDIX B - PROGRAMS FOR (6), (10), AND (13B)

Table B-1. Program for (6)

```

10  M=7                ! > 0
20  Gamma=1.31         ! > 0
30  DOUBLE M,Ms        ! INTEGERS
40  S=0.
50  IF (M MODULO 2)=1 THEN 110
60  FOR Ms=1 TO M-1 STEP 2
70  S=S+FNG(Gamma*Ms)
80  NEXT Ms
90  V1=2.*S
100 GOTO 150
110 FOR Ms=2 TO M-1 STEP 2
120 S=S+FNG(Gamma*Ms)
130 NEXT Ms
140 V1=FNG(0.)+2.*S
150 PRINT M,Gamma,V1
160 END
170 !
180 DEF FNG(W)

```

Table B-2. Program for (10)

```

10  M=6                ! > 0
20  Gamma=.71          ! > 0
30  DOUBLE M,Ms        ! INTEGERS
40  G2=2.*Gamma
50  S=0.
60  FOR Ms=1 TO M-1
70  S=S+(M-Ms)*FNG(G2*Ms)
80  NEXT Ms
90  V2=M*FNG(0.)+2.*S
100 PRINT M,Gamma,V2
110 END
120 !
130 DEF FNG(W)

```

Table B-3. Program for (13B)

```

10  M=9                      !  > 0
20  Gamma=.79                !  > 0
30  DOUBLE M,Ms,Ks          !  INTEGERS
40  DIM W(100)
50  REDIM W(1:M)
60  CALL Weights(M,W(.*))    !  REAL WEIGHTS
70  G2=2.*Gamma
80  S=0.
90  FOR Ms=1 TO M-1
100 Phi=0.
110 FOR Ks=Ms+1 TO M
120 Phi=Phi+W(Ks)*W(Ks-Ms)  !  CORRELATION OF WEIGHTS
130 NEXT Ks
140 S=S+Phi*FNGr(G2*Ms)
150 NEXT Ms
160 Phi=0.
170 FOR Ks=1 TO M
180 Phi=Phi+W(Ks)*W(Ks)
190 NEXT Ks
200 V4=Phi*FNGr(0.)+2.*S
210 PRINT M,Gamma,V4
220 END
230 !
240 SUB Weights(DOUBLE M,REAL W(.*))
250 DOUBLE Ks                !  INTEGER
260 T=2.*PI/M
270 FOR Ks=1 TO M
280 D=Ks-.5
290 W(Ks)=1.                 !  FLAT WEIGHTS
300 W(Ks)=.5-.5*COS(T*D)     !  HANN WEIGHTS
310 W(Ks)=.54-.46*COS(T*D)  !  HAMMING WEIGHTS
320 NEXT Ks
330 MAT W=W/SUM(W)          !  NORMALIZATION
340 SUBEND
350 !
360 DEF FNGr(W)

```

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